3-5 有熱源的問題

有時候，材料內某些型式的能量會轉變爲熱能，如電能、化學能及核能等，此時，可視爲內部有熱源。當吾人處理一維、穩態且有熱源的問題時，其統御方程式（熱擴散方程式）爲

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(kr^n\frac{\partial T}{\partial r}\right)+\dot{g}=0 \qquad (3\text{-}5.1)$$

其中 $n=\begin{cases}0：直角座標\\1：圓柱座標\\2：球座標\end{cases}$

$\dot{g}$ 爲單位體積的熱源。

若 k 爲常數，則上式變成

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=0 \qquad (3\text{-}5.2)$$

吾人以直角及圓柱座標分別討論如下：

1. 平面牆

（3-5.2）變成

$$\frac{d}{dx}\left(\frac{dT}{\partial x}\right)+\frac{\dot{g}}{k}=0$$

積分

$$\frac{dT}{dx}=\int\frac{-\dot{g}}{k}dx$$

再積分

$$T=\int\left(\int\frac{-\dot{g}}{k}dx\right)dx$$

若 $\dot{g}$ 爲常數，則溫度分布之通解爲

$$T=-\frac{\dot{g}}{2k}x^2+c_1x+c_2 \quad (3\text{-}5.3a)$$

將 B.C's 代入，即可求得 c_1 與 c_2，且 $\dot{q}_x$ 可得

$$\dot{q}_x=-k\frac{dT}{dx}=\dot{g}x-c_1k \quad (3\text{-}5.3b)$$

2. 圓柱（管長 L）

（3-5.2）變成

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{g}}{k}=0$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{\dot{g}r}{k}$$

積分

$$r\frac{dT}{dr}=\int\frac{-\dot{g}r}{k}dr$$

$$\Rightarrow \frac{dT}{dr}=\frac{1}{r}\int-\frac{\dot{g}r}{k}dr$$

再積分

$$T=\int\left(\frac{1}{r}\int-\frac{\dot{g}r}{k}dr\right)dr$$

若 $\dot{g}$ 爲常數，則溫度分布之通解爲

$$T=-\frac{\dot{g}r^2}{4k}+c_1\ell\text{n}r+c_2 \quad (3\text{-}5.4)$$

將 B.C's 代入，即可求得 c_1 與 c_2。對實心圓柱而言，當 r 趨近於 0 時，ℓnr 趨近於 $-\infty$，故 $c_1 = 0$，上式變成

$$T = -\frac{\dot{g}r^2}{4k} + c_2 \tag{3-5.5a}$$

而 $\dot{q}_r$ 及 $\dot{Q}_r$ 分別為

$$\dot{q}_r = -k\frac{dT}{dr} = \frac{\dot{g}r}{2} \tag{3-5.5b}$$

$$\dot{Q}_r = \dot{q}_r(2\pi rL) = \dot{g}\pi r^2 L = \dot{g}\forall_r \tag{3-5.5c}$$

其中 $\forall_r$ 為半徑 r 之圓柱體積。

例題3

Consider the plane wall with uniformly distributed heat sources shown in figure. The thickness of the wall in the x direction is $2L = 2$ m, A current passing through this conducting material with the heat generated per unit volume is $\dot{q} = 200$ W/m^3, and assume the thermal conductivity $k = 10$ W/m·K does not vary with temperature, the differential equation which governing the heat flow is

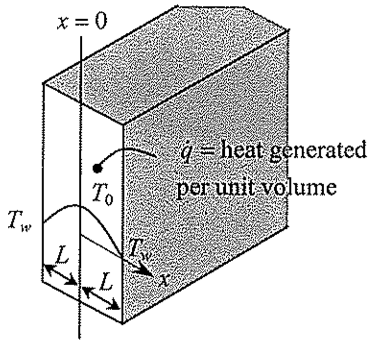
$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

For boundary conditions either side of the wall temperature, $T_w = 30°\text{C}$ at $x = +L, -L$.

Determine:

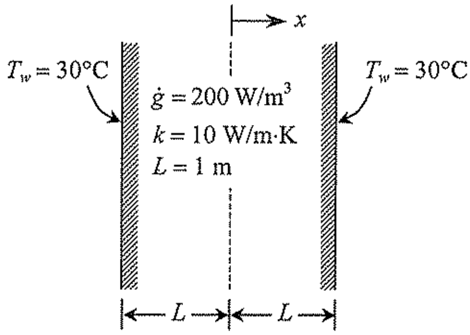
(1)the temperature of the midplane, T_0

(2)the temperature distribution, $\dfrac{(T - T_w)}{(T_0 - T_w)}$ (15%)



【中華機械】

解：



由題意可知，熱擴散方程式為

$$\frac{d^2T}{dx^2}+\frac{\dot{g}}{k}=0$$

積分兩次可得通解為

$$T(x)=-\frac{\dot{g}}{2k}x^2+c_1x+c_2$$

而 B.C's 為

$$\begin{cases} T(-L)=T_w \\ T(L)=T_w \end{cases}$$

故 c_1 及 c_2 分別為

$$c_1 = 0$$

$$c_2 = \frac{\dot{g}L^2}{2k} + T_w$$

因此，溫度分布為

$$T(x) = \frac{\dot{g}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_w$$

(1)　$$T_0 = T(x=0) = \frac{\dot{g}L^2}{2k} + T_w = \frac{200(1)^2}{2(10)} + 30$$

$$= 40 \quad (°\text{C})$$

(2)由上兩式可得

$$\frac{T - T_w}{T_0 - T_w} = \frac{\frac{\dot{g}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right)}{\frac{\dot{g}L^2}{2k}} = 1 - \frac{x^2}{L^2} = 1 - \frac{x^2}{(1)^2}$$

$$= 1 - x^2$$

例題4

A plane wall of thickness 0.1 m and thermal conductivity 25 W/(m·K) having uniform volumetric heat generation of 0.3 MW/m^3 is insulated on one side, while the other side is exposed to a fluid at 92°C. The convection heat transfer coefficient between the wall and the fluid is 500 W/(m^2·K). Determine the maximum temperature in the wall. 【台科大機械】