

$$\begin{aligned}
 [\text{詳解}] \iiint_D z dV &= \iint_{\text{看下面}} \int_{z=0}^{z=l-x-y} z dz dy dx \\
 &= \int_{x=0}^{x=l} \int_{y=0}^{y=l-x} \int_{z=0}^{z=l-x-y} z dz dy dx = \int_{x=0}^{x=l} \int_{y=0}^{y=l-x} \frac{1}{2} (l-x-y)^2 dy dx \\
 &= \int_{x=0}^{x=l} \left[-\frac{1}{6} (l-x-y)^3 \right]_{y=0}^{y=l-x} dx = \int_{x=0}^{x=l} \frac{1}{6} (l-x)^3 dx = \frac{1}{24}
 \end{aligned}$$

題型 2 圓柱座標

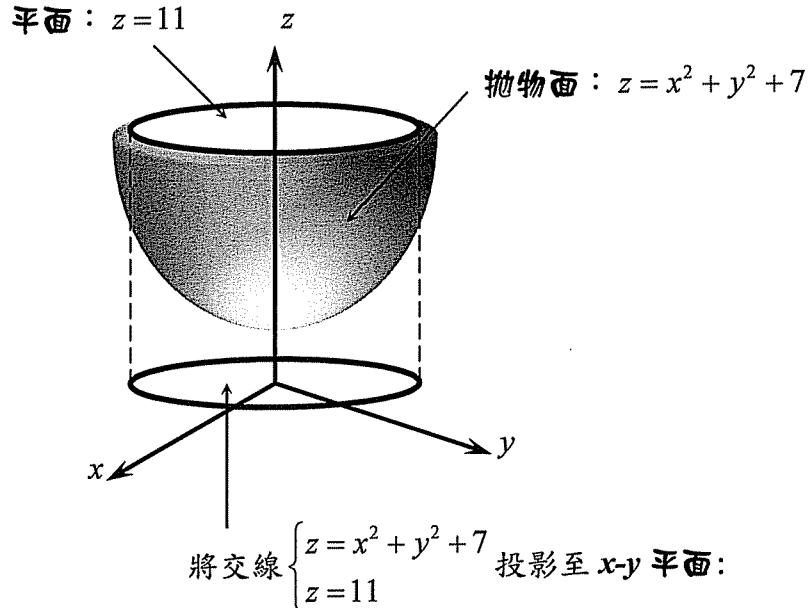
範例 3

Find the volume of the region bounded by the **paraboloid** $z = x^2 + y^2 + 7$

and the **plane** $z = 11$.

(10%)【96 政大】

【詳解】



$$x^2 + y^2 + 7 = 11 \Rightarrow x^2 + y^2 = 4$$

$$\begin{aligned}
V &= \iint_{\text{看下面}} \left(\int_{z=x^2+y^2+7}^{z=11} dz \right) dx dy = \iint_{\text{看下面}} [11 - (x^2 + y^2 + 7)] dx dy \\
&= \iint_{\text{看下面}} (4 - x^2 - y^2) dx dy = \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx \\
&= 4 \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx \\
&= 4 \int_{x=0}^{x=2} \left[(4 - x^2)y - \frac{1}{3}y^3 \right]_{y=0}^{y=\sqrt{4-x^2}} dx \\
&= 4 \int_{x=0}^{x=2} \left[(4 - x^2)\sqrt{4 - x^2} - \frac{1}{3}(\sqrt{4 - x^2})^3 \right] dx \\
&= 4 \int_{x=0}^{x=2} \frac{2}{3}(\sqrt{4 - x^2})^3 dx
\end{aligned}$$

令 $x = 2 \sin \theta$ ，則 $dx = 2 \cos \theta d\theta$

此時 $(\sqrt{4 - x^2})^3 = 8 \cos^3 \theta$

$$\begin{cases} \text{當 } x = 2 \rightarrow \theta = \frac{\pi}{2} \\ \text{當 } x = 0 \rightarrow \theta = 0 \end{cases}$$

$$\begin{aligned}
\text{代入 } V &= 4 \int_{x=0}^{x=2} \frac{2}{3}(\sqrt{4 - x^2})^3 dx = \frac{8}{3} \int_{\theta=0}^{\theta=\frac{\pi}{2}} 8 \cos^3 \theta \cdot 2 \cos \theta d\theta \\
&= \frac{128}{3} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos^4 \theta d\theta = \frac{128}{3} \left(\frac{1 \cdot 3}{2 \cdot 4} \frac{\pi}{2} \right) = 8\pi
\end{aligned}$$

真是太難積了，改用圓柱座標！

$$\text{【詳解】令圓柱座標 : } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

$$\begin{aligned}
 V &= \iint_{\text{看下面}} \left(\int_{z=x^2+y^2+7}^{z=11} dz \right) dx dy = \iint_{\text{看下面}} \int_{z=r^2+7}^{z=11} r dz dr d\theta \\
 &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(\int_{z=r^2+7}^{z=11} dz \right) r dr d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4-r^2) r dr d\theta \\
 &= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{r=0}^{r=2} (4r - r^3) dr = 8\pi
 \end{aligned}$$

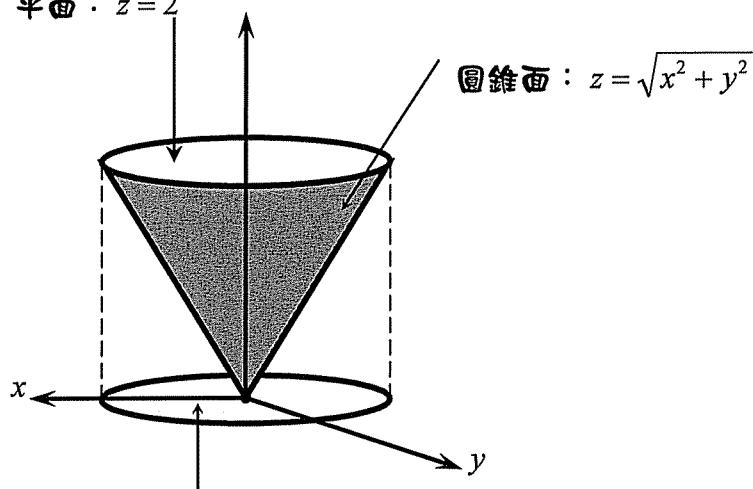
範例 4

Convert the integral $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 dz dy dx$

to an equivalent integral in **cylindrical coordinates**.

Answer: _____ (Do **not** evaluate the integral). (8%) 【97 台聯大】

【詳解】

平面 : $z = 2$ 圓錐面 : $z = \sqrt{x^2 + y^2}$

將交線 $\begin{cases} z = \sqrt{x^2 + y^2} \\ z = 2 \end{cases}$ 投影至 $x-y$ 平面 $\Rightarrow x^2 + y^2 = 4$