範例導引，一路領先

範例 1

推導**傅立葉級數(Fourier series)**的公式：

$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L})$ 其中

$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx$, $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx$, $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$,

$f(x)$ 有何**限制條件**？ (15%)【95 成大資源】

【詳解】**限制條件**：將 $f(x)$ 視為週期 $\boxed{T=2L}$ 的**週期性函數**。

推導：

令 $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x)$

$$=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)$$

① $$a_0=\frac{<f(x),1>}{\|1\|^2}=\frac{\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)dx}{\int_{-\frac{T}{2}}^{\frac{T}{2}}1^2dx}=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)dx=\frac{1}{2L}\int_{-L}^{L}f(x)dx$$

② $$a_n=\frac{<f(x),\cos\frac{2n\pi}{T}x>}{\left\|\cos\frac{2n\pi}{T}x\right\|^2}=\frac{\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos\frac{2n\pi}{T}xdx}{\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos^2\frac{2n\pi}{T}xdx}$$

$$=\frac{1}{T/2}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos\frac{2n\pi}{T}xdx=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx$$

③ $$b_n=\frac{<f(x),\sin\frac{2n\pi}{T}x>}{\left\|\sin\frac{2n\pi}{T}x\right\|^2}=\frac{\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin\frac{2n\pi}{T}xdx}{\int_{-\frac{T}{2}}^{\frac{T}{2}}\sin^2\frac{2n\pi}{T}xdx}$$

$$=\frac{1}{T/2}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin\frac{2n\pi}{T}xdx=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx$$

範例 2

Periodic function $f(x)$ has **period** 2π and $f(x)=x+\pi \quad -\pi<x<\pi$

(a) find the **Fourier series** of $f(x)$.

(b) find the value of this series at $x=\pi$.

(c) use the result above to show that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.

【95 交大土木、94 成大醫工、中央機械、中央光機電、中央能源、92 北科機電、北科高分子、91 逢甲土木、90 台大應力、台大環工】

【詳解】(a) $f(x)=a_0+\sum_{n=1}^{\infty}\{a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x\}$

$$=a_0+\sum_{n=1}^{\infty}\{a_n\cos nx+b_n\sin nx\}$$

$$a_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x+\pi)dx=\pi$$

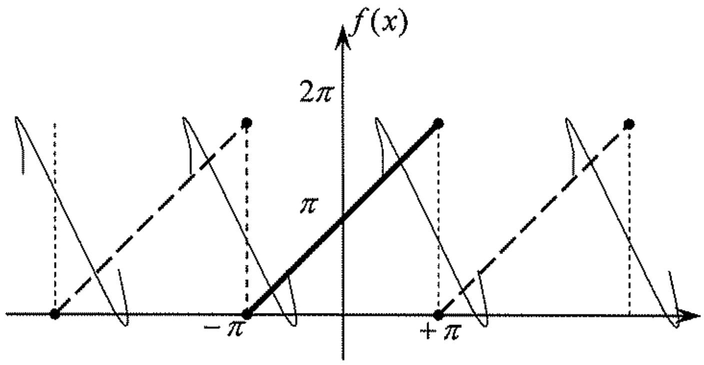
$$a_n=\frac{1}{T/2}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nxdx=0$$

$$b_n=\frac{1}{T/2}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nxdx$$

$$=-\frac{2}{n}\cos n\pi=\frac{2(-1)^{n+1}}{n}$$

$$\Rightarrow \quad f(x)=\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

(b) 由 **Dirichlet 收斂條件**



當 $x=\pi$： $\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi \overset{\text{收斂至}}{=} \frac{1}{2}[f(\pi^{+})+f(\pi^{-})]=\pi$

(c) 當 $x=\frac{\pi}{2}$： $\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2} \overset{\text{收斂至}}{=} f(\frac{\pi}{2})=\frac{3\pi}{2}$

$$\Rightarrow 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}=\frac{\pi}{2} \quad \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}=\frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

範例 3

(a) Obtain the **Fourier series representation** of the function $f(x)$,

$$f(x+4)=f(x) \text{ and } f(x)=\begin{cases} x+2 & \text{for } -2<x<0 \\ 1 & \text{for } 0<x<2 \end{cases}.$$

(b) Calculate $\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}$. 【成大電機、中山環工】