19-6 Stoke 定理(Stoke's Theorem)

⇨重點整理⇦

1. 空間 **Stoke** 定理：

D 為第 II 類單連通區間，

$\vec{f}(x,y,z)=f_1\vec{i}+f_2\vec{j}+f_3\vec{k}$

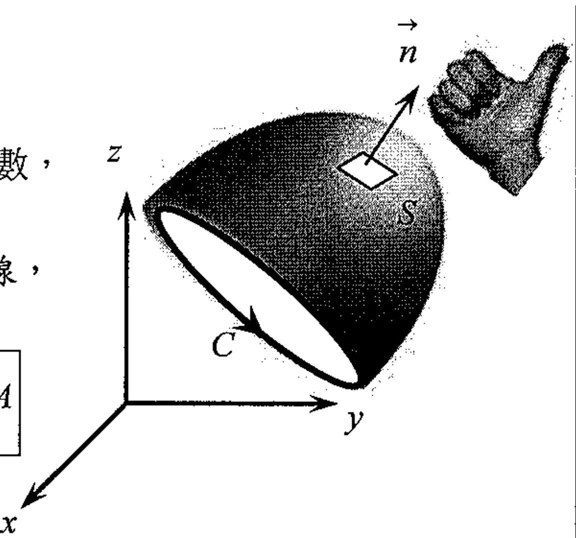
在 *D* 內為一階連續向量函數，

S 為 *D* 內之一曲面，

C 為 *S* 上之一微量封閉曲線，

A 為 *C* 內部之面積，

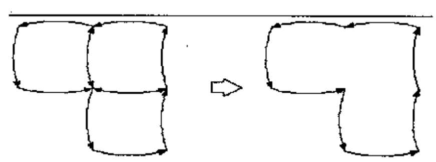
則 $\oint_C \vec{f}\cdot d\vec{r}=\iint_S(\nabla\times\vec{f})\cdot\vec{n}\,dA$



證明 $\textbf{curl}\vec{f}=\nabla\times\vec{f}=(\lim_{A\to0}\frac{1}{A}\oint_C\vec{f}\cdot d\vec{r})\vec{e}_{\max}=(\lim_{A\to0}\frac{1}{A}\oint_C\vec{f}\cdot d\vec{r})\vec{n}$

$$(\nabla\times\vec{f})\cdot\vec{n}=\lim_{A\to0}\frac{1}{A}\oint_C\vec{f}\cdot d\vec{r}\approx\frac{1}{\Delta A}\oint_{\Delta C}\vec{f}\cdot d\vec{r}$$

$$(\nabla\times\vec{f})\cdot\vec{n}\Delta A\approx\oint_{\Delta C}\vec{f}\cdot d\vec{r}$$



$$\sum_i(\nabla\times\vec{f})\cdot\vec{n}_i\Delta A_i\approx\sum_i\oint_{\Delta C_i}\vec{f}\cdot d\vec{r}$$

當 $\Delta A_i\to0$：$\lim_{\Delta A_i\to0}\sum_i(\nabla\times\vec{f})\cdot\vec{n}_i\Delta A_i\approx\lim_{\Delta C_i\to0}\sum_i\oint_{\Delta C_i}\vec{f}\cdot d\vec{r}$

由圖知相鄰部份會相抵消，最後只剩下最外表部份

$$\iint_S(\nabla\times\vec{f})\cdot\vec{n}\,dA\approx\oint_C\vec{f}\cdot d\vec{r}$$

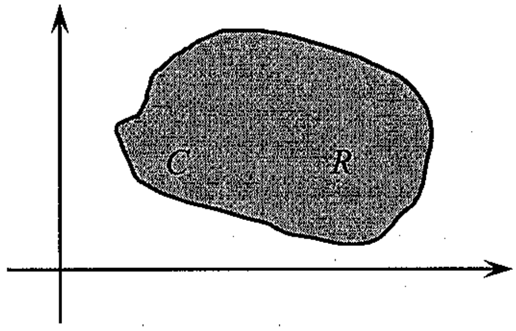
2. 平面 **Stoke** 定理：

R 為單連通區間，且 $\vec{f}(x,y)=f_1\vec{i}+f_2\vec{j}$

在 R 內為一階連續向量函數，

C 為 R 內之一微量封閉曲線，

則 $\oint_c \vec{f}\cdot d\vec{r}=\iint_R(\nabla\times\vec{f})\cdot\vec{k}dA$



3. **Stoke** 定理之相關定理：

D 為第 II 類單連通區間，$\vec{f}(x,y,z)=f_1\vec{i}+f_2\vec{j}+f_3\vec{k}$

在 D 內為一階連續向量函數，

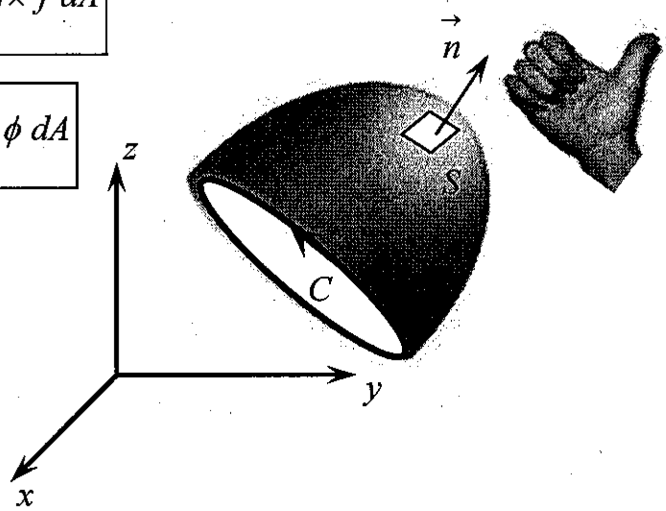
S 為 D 內之一曲面，

C 為 S 上之一微量封閉曲線，則

(1) **Stoke 定理**：$\oint_C \vec{f}\cdot d\vec{r}=\iint_S(\nabla\times\vec{f})\cdot\vec{n}\,dA$ ⇨ $\oint_C \vec{e}_t\cdot\vec{f}\,ds=\iint_S(\vec{n}\times\nabla)\cdot\vec{f}\,dA$

(2) $\oint_C \vec{e}_t\times\vec{f}\,ds=\iint_S(\vec{n}\times\nabla)\times\vec{f}\,dA$

(3) $\oint_C \vec{e}_t\,\phi\,ds=\iint_S(\vec{n}\times\nabla)\ \phi\,dA$



範例導引，一路領先

題型 1　3-D Stoke 定理

範例 1

Give the vector $\vec{A}=(x^2-y^2)\vec{i}+2xy\vec{j}$.

(a) Evaluate the **surface integral** $\int(\nabla\times\vec{A})\cdot d\vec{\sigma}$ over a rectangle in the (x,y) plane bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$,

Here, $d\vec{\sigma}$ is a **surface element vector.**

(b) Evaluate the **line integral** $\oint\vec{A}\cdot d\vec{l}$ around the boundary of the rectangle.

Here, $d\vec{l}$ is a **line element vector.**

(c) Compare the results you have obtained for (a) and (b). What **theorem** of vector analysis is relevant to the conclusion you draw from the comparison? (20%)【95 台大物理】

【詳解】(a) 已知 $\vec{A}=(x^2-y^2)\vec{i}+2xy\vec{j}$

$$\Rightarrow\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2-y^2 & 2xy & 0\end{vmatrix}=4y\vec{k}$$

$$\Rightarrow\int\nabla\times\vec{A}\cdot d\vec{\sigma}=\int 4y\vec{k}\cdot\vec{k}\,dxdy=\int 4y\ dxdy=\int_0^b\int_0^a 4ydxdy=2ab^2$$

(b) $\oint\vec{A}\cdot d\vec{l}=\oint_C\left[(x^2-y^2)dx+2xydy\right]$

由 **Green theorem**

上式 $= \iint\left[\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(x^2-y^2)\right]dxdy = \int_0^b\int_0^a 4ydxdy = 2ab^2$

(c) 比較(a) (b)，得：$\int \nabla\times\vec{A}\cdot d\vec{\sigma} = \oint \vec{A}\cdot d\vec{l}$

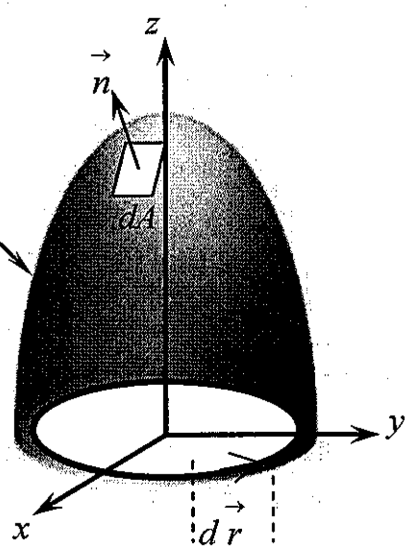
符合 **Stoke's theorem** 的結論，本題得證。

範例 2

Verify the **Stoke's theorem**: $\iint \nabla\times\vec{F}\cdot\vec{n}dA = \oint \vec{F}\cdot d\vec{r}$ (see Fig.)

where the vector function : $\vec{F} = y\vec{i} - z\vec{j} + 3x\vec{k}$;

surface $S: z = f(x,y) = 4-(x^2+y^2)$, $z\geq 0$



(20%)【95 成大工科】

【詳解】(1) $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -z & 3x \end{vmatrix} = \vec{i} + 3\vec{j} - \vec{k}$

令 $\phi(x,y,z) = x^2+y^2+z-4 = 0$

⇨ $\nabla\phi(x,y,z) = 2x\vec{i} + 2y\vec{j} + \vec{k}$

⇨ $\vec{n}dA = \nabla\phi\frac{dxdy}{|\nabla\phi\cdot\vec{k}|} = (2x\vec{i}+2y\vec{j}+\vec{k})dxdy$