⑵ $\because P(\bar{X} \ge k' \mid \mu = 75) = 0.1 \Rightarrow P\left(Z \ge \dfrac{k'-75}{10/\sqrt{25}}\right) = 0.1$

$\therefore \dfrac{k'-75}{10/\sqrt{25}} \doteq 1.28 \Rightarrow k' = 77.56$

故知檢定之一致最佳危險域為 $C = \{\bar{x} \mid \bar{x} \ge 77.56\}$。

難易度：🏆🏆🏆🏆

**例題 5.4**

設 $(X_1, X_2, \ldots, X_n)$ 為抽自母體 $f(x) = \dfrac{1}{\theta} e^{-\frac{x}{\theta}}$，$x > 0$ 之一組隨機樣本，試求檢定 $H_0$：$\theta = \theta_0$，$H_1$：$\theta = \theta_1$（$\theta_1 > \theta_0$）且檢定大小為 $\alpha$ 的最強力檢定（MPT）。（交大管科、財金、台大財金）

➠因 $L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \dfrac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}}$，故知

$$\frac{L(\theta_0)}{L(\theta_1)} = \frac{\dfrac{1}{\theta_0^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta_0}}}{\dfrac{1}{\theta_1^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta_1}}} = \left(\frac{\theta_1}{\theta_0}\right)^n e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i}$$

令 $\dfrac{L(\theta_0)}{L(\theta_1)} \le k \Rightarrow \left(\dfrac{\theta_1}{\theta_0}\right)^n e^{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i} \le k \Rightarrow n\ell n\left(\dfrac{\theta_1}{\theta_0}\right) - \left(\dfrac{1}{\theta_0} - \dfrac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i \le k$

$$\Rightarrow -\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i \le k - n\ell n\left(\frac{\theta_1}{\theta_0}\right) \Rightarrow \sum_{i=1}^{n} x_i \ge \frac{k - n\ell n\left(\frac{\theta_1}{\theta_0}\right)}{-\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)} = k'$$

即 $\dfrac{L(\theta_0)}{L(\theta_1)} \le k \overset{\text{相當於}}{\Rightarrow} \sum_{i=1}^{n} x_i \ge k'$，故由Neyman-Pearson引理知

$C = \{\sum_{i=1}^{n} x_i \mid \sum_{i=1}^{n} x_i \ge k'\}$ 為此檢定之最佳危險域，其中 $k'$ 由 $\alpha$ 決定

又 $\dfrac{2\sum_{i=1}^{n}X_i}{\theta_0}\xrightarrow{H_0\text{爲真下}}\chi^2(2n)$ ，故

$$P(\sum_{i=1}^{n}X_i\ge k'\mid\theta=\theta_0)=\alpha\Rightarrow P\left(\frac{2\sum_{i=1}^{n}X_i}{\theta_0}\ge\frac{2k'}{\theta_0}\middle|\theta=\theta_0\right)=\alpha$$

$$\therefore k'=\frac{\theta_0}{2}\chi_\alpha^2(2n)$$

難易度：🏆🏆🏆🏆

**●例題 5.5●**

設r.v. $X$ 之機率分配 $f(x)=1+\theta^2\left(x-\dfrac{1}{2}\right)$ ，$0\le x\le 1$，$0\le\theta\le\sqrt{2}$

⑴ 今自此分配 $f(x)$ 抽出一個觀察值 $X$ ，試求檢定 $H_0:\theta=0$ 及 $H_1:\theta>0$ 且 $\alpha=0.1$ 之最佳危險域。

⑵試問⑴是否為一致最強力檢定。　（中央財管）

➠⑴ ∵令 $\theta_1$ 爲大於0之任一數，又

$$\frac{L(0)}{L(\theta_1)}=\frac{1}{1+\theta_1^2\left(x-\frac{1}{2}\right)}$$

令 $\dfrac{L(0)}{L(\theta_1)}\le k\Rightarrow\dfrac{1}{1+\theta_1^2\left(x-\frac{1}{2}\right)}\le k$

$$\Rightarrow\theta_1^2\left(x-\frac{1}{2}\right)\ge\frac{1}{k}-1$$

$$\Rightarrow x\ge\frac{1}{\theta_1^2}\left(\frac{1}{k}-1\right)+\frac{1}{2}=k'$$

$$\therefore\frac{L(0)}{L(\theta_1)}\le k\xrightarrow{\text{相當於}}X\ge k'$$

又 $P(X\ge k'\mid\theta=0)=0.1\Rightarrow\int_{k'}^{1}1dx=0.1\Rightarrow k'=0.9$

故最佳危險域爲 $C=\{x\mid x>0.9\}$

⑵因 $\theta_1$ 爲大於0之任一數，故此危險域$C$對 $H_1$ 任一點皆成立，故爲一致最強力檢定。