

115 台大健康數據拓析統計所

科目：基礎數學(微積分、線性代數)

📖 題目

1. 如果 $\mathbf{P} = \mathbf{P}^T \mathbf{P}$ ，證明 $\mathbf{P}$ 是一個投影矩陣。
2. 請求出投影矩陣 $\mathbf{P}$ ，其投影空間為由 $\mathbf{a}_1 = (1, 1, 0)$, $\mathbf{a}_2 = (1, 1, 1)$ 所展出的子空間，並找出一個非零向量 $\mathbf{b}$ ，使得在此投影下被投影為零向量(即 $\mathbf{P}\mathbf{b} = \mathbf{0}$)。

3. 求 $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$ 的行列式 $\det(\mathbf{A})$ 。計算 $\det(2\mathbf{A})$, $\det(-\mathbf{A})$, $\det(\mathbf{A}^2)$ ，跟 $\det(\mathbf{A}^{-1})$ 。

4. 求 $\int \frac{\cos x dx}{\sqrt[7]{\sin^{11} x}}$ 。

5. 求 $\int e^x e^{e^x} dx$ ， e 是自然對數的底數。

6. 求 $\int \frac{\ln x}{x} dx$ 。

7. 求 $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2}$ ， e 是自然對數的底數。

8. 求 $\lim_{x \rightarrow \infty} \frac{x^n}{e^x}$ ， e 是自然對數的底數。

9. $f(x) = \frac{x(x^2 - 5) + 5(x - 1)}{x^3 - 1}$ ，請求出 $f'(x)$ 。

10. $f(x) = (x^3 + x)^4$ ，請求出 $f''(x)$ 。

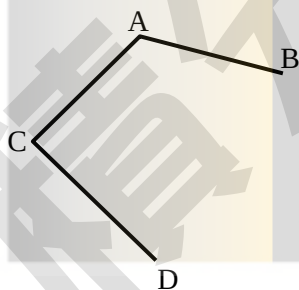
11. $\lim_{x \rightarrow -2} \frac{x+2}{(3x^2 + 10x + 8)^2}$ ，請求出此式解。

12. $x = 1$, $x = 3$, $y = x^2 - 1$, $y = -3x + 9$ ，請繪製出四條線圍起來的圖形並計算其面積。

13. 矩陣 $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 4 \\ 1 & -2 & -2 \end{bmatrix}$ ，請求出矩陣 A 的特徵值(eigenvalue)及特

徵向量(eigenvector)，並計算 A^9 。

14. 一傳染病原於 C 鄉鎮大幅出現，其特色為經過一小時後，該病原會隨機的轉往鄰近鄉鎮(會停留於原本鄉鎮)，請寫出鄉鎮間的轉移矩陣，若起始時 C 鄉鎮具有 10000 隻病原，請問在經過極長時間後，各鄉鎮具有的病原數目各為多少？



解答

1. $\mathbf{P}^2 = \mathbf{P} \rightarrow \mathbf{P}\mathbf{P} = \mathbf{P} \rightarrow \mathbf{P}^T\mathbf{P} = \mathbf{P}$ (故得證)。

$$2. \text{ 令 } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\therefore \mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{令 } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \text{ 則 } \mathbf{P}\mathbf{b} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(b_1 + b_2) \\ \frac{1}{2}(b_1 + b_2) \\ b_3 \end{bmatrix} = \mathbf{0}$$

$$\text{得 } b_2 = -b_1, b_3 = 0, \therefore \mathbf{b} = \begin{bmatrix} b_1 \\ -b_1 \\ 0 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}。$$

$$3. \det(\mathbf{A}) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -\frac{1}{2} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -\frac{1}{2} + 1 = \frac{1}{2}。$$

$$\det(2\mathbf{A}) = (2)^4 \cdot \frac{1}{2} = 8; \quad \det(-\mathbf{A}) = (-1)^4 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\det(\mathbf{A}^2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}; \quad \det(\mathbf{A}^{-1}) = \frac{1}{\frac{1}{2}} = 2。$$

$$4. \int \frac{\cos x dx}{\sqrt[7]{\sin^{11} x}} = -\frac{7}{4} (\sin x)^{-\frac{4}{7}} + C。$$

115 中央統研所

科目：基礎數學

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1. Find the minimize of $\sum_{i=1}^n x_i$ subject to $\sum_{i=1}^n x_i^2 \leq 1$ or $\sum_{i=1}^n (x_i + 1)^2 \leq 1$.

2. Let $\mathbf{x} \in \mathbb{R}^p$ be a nonzero vector.

(a) Derive the largest eigenvalue of the matrix $\mathbf{\Omega} = (\mathbf{I}_p + \mathbf{x}\mathbf{x}^T)^{-1}$ and its corresponding eigenvector, where $\mathbf{I}_p$ denotes the $n \times n$ identity matrix.

(b) Express an explicit expression for $\mathbf{\Omega}$ in terms of $\mathbf{x}$.

3. Find the value of p , for which the following limits exist.

(a) $\lim_{x \rightarrow 0^+} x^p \sin \frac{1}{x}$

(b) $\lim_{x \rightarrow \infty} x^p \sin \frac{1}{x}$

4. For $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m \times n}$, the vectorization operator $\text{vec}(\cdot)$ is defined by

$$\text{vec}(\mathbf{A}) = (a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn})^T \in \mathbb{R}^{mn}$$

Show that $\frac{\partial}{\partial \text{vec}(\mathbf{A})} \text{tr}(\mathbf{A}\mathbf{A}^T) = 2\text{vec}(\mathbf{A})$

In addition, for $\mathbf{B} = (b_{ij}) \in \mathbb{R}^{n \times q}$, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{nm \times nq}$$

Show that $\text{vec}(\mathbf{AB}) = (\mathbf{I}_q \otimes \mathbf{A})\text{vec}(\mathbf{B})$.

5. Calculate the following limits.

(a) $\lim_{b \rightarrow \infty} \int_0^b e^{-xy} \sin x dx$ for $y > 0$. (b) $\lim_{b \rightarrow \infty} \int_0^b e^{-xy} \frac{\sin x}{x} dx$ for $y > 0$.

6. Let matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a skew-symmetric, that is $\mathbf{A}^T = -\mathbf{A}$. Show that

$\det(\mathbf{A}) = 0$ if n is odd. What if n is even?

7. Let f be a positive continuous function in $[0,1]$. Let M denote the maximum value of f on $[0,1]$. Calculate the following limit.

$$\lim_{n \rightarrow \infty} \left\{ \int_0^1 \left(\frac{f(x)}{M} \right)^n dx \right\}^{1/n}$$

解答

1.

(a) 利用 Cauchy-Schwarz 不等式

$$\left| \sum_{i=1}^n x_i \right| \leq \sqrt{\sum_{i=1}^n x_i^2} \sqrt{\sum_{i=1}^n 1} = 1 \cdot \sqrt{n} = \sqrt{n}$$

$\therefore \sum_{i=1}^n x_i$ 的最小值為 $-\sqrt{n}$ 。

(b) 令 $u_i = x_i - 1$ 即為 (a) 之情況

$\therefore \sum_{i=1}^n x_i$ 的最小值為 $-\sqrt{n} - n$ 。

2. $\mathbf{x} \in \mathbb{R}^p$

(a) $\mathbf{\Omega} = (\mathbf{I}_p + \mathbf{x}\mathbf{x}^T)^{-1}$

令 $\mathbf{A} = \mathbf{I} + \mathbf{x}\mathbf{x}^T$

由 $\mathbf{Ax} = \mathbf{Ix} + \mathbf{x}\mathbf{x}^T\mathbf{x} = \mathbf{x} + \mathbf{x} = 2\mathbf{x}$ 知 $\mathbf{x}$ 為特徵向量，2 為特徵值。

若 $\mathbf{u}$ 為與 $\mathbf{x}$ 正交之任意向量，即有 $\mathbf{u}^T\mathbf{x} = 0$

則 $\mathbf{Au} = \mathbf{Iu} + \mathbf{x}\mathbf{x}^T\mathbf{u} = \mathbf{u}$ ，知 $\mathbf{u}$ 為特徵向量，1 為特徵值

115 台北大學統計所

科目：基礎數學

📖 題目

一、Calculus

1. Find the limit for $\lim_{x \rightarrow \pi} \left(\csc \frac{x}{2} \right)^{\tan^2 \frac{x}{2}} = ?$

2. Find the limit for $\lim_{n \rightarrow \infty} \left(\frac{\pi}{\sqrt{n^2+1}} + \frac{\pi}{\sqrt{n^2+2^2}} + \cdots + \frac{\pi}{\sqrt{n^2+n^2}} \right) = ?$

3. Find the slope of the tangent line to the curve

$$2(x^2 + y^2)^2 + (x^2 - 1) = 25(x^2 - y^2) + 2^x \text{ at point } (3,1).$$

4. Find the indefinite integral $\int \frac{1}{\sqrt{4+e^x}} dx = ?$

5. Find the definite integral $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} dy dx = ?$

二、Linear Algebra

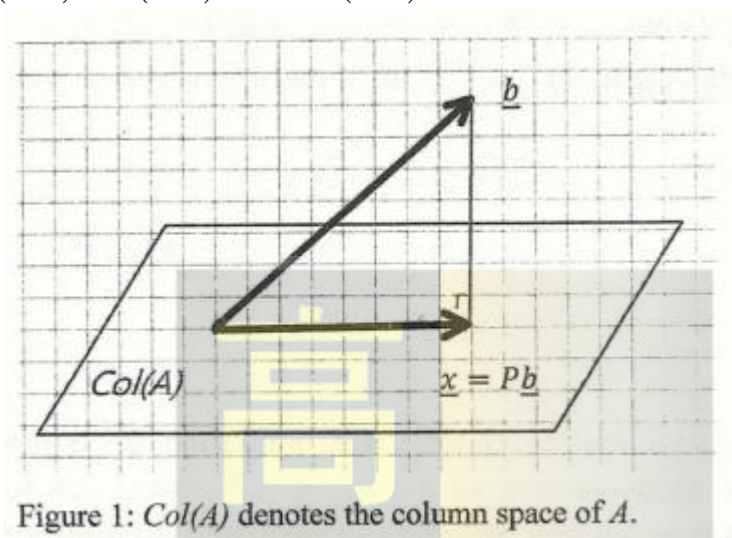
1. Suppose $\mathbf{x}$ and $\mathbf{y}$ are two linear independent vectors in $\mathbb{R}^n$, and let U be the span of $\mathbf{x}$ and $\mathbf{y}$, that is $U = \text{span}\{\mathbf{x}, \mathbf{y}\}$. Let $\mathbf{A} = \mathbf{xy}^T + \mathbf{yx}^T$

Prove that the null space $N(\mathbf{A})$ is orthogonal to U , meaning every $\mathbf{v}$ in $N(\mathbf{A})$ is orthogonal to every $\mathbf{w}$ in U .

2. Let $\mathbf{A}$ be an $m \times n$ matrix with independent columns, and let $\mathbf{A} = \mathbf{QR}$ be its QR factorization. The orthogonal projection of any vector $\mathbf{b} \in \mathbb{R}^m$ onto the column space of $\mathbf{A}$ can be represented by $\mathbf{x} = \mathbf{Pb}$, where $\mathbf{P}$ is the projection matrix and $\mathbf{x} \in \text{Col}(\mathbf{A})$ as shown in Figure 1.

(a) Show that $\mathbf{P} = \mathbf{QQ}^T$.

(b) Let $\mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \cdots \ \mathbf{q}_n]$. Show that $\mathbf{x}$ can be written as a linear combination of $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n$ and represented by

$$\mathbf{x} = (\mathbf{q}_1^T \mathbf{b}) \mathbf{q}_1 + (\mathbf{q}_2^T \mathbf{b}) \mathbf{q}_2 + \cdots + (\mathbf{q}_n^T \mathbf{b}) \mathbf{q}_n.$$


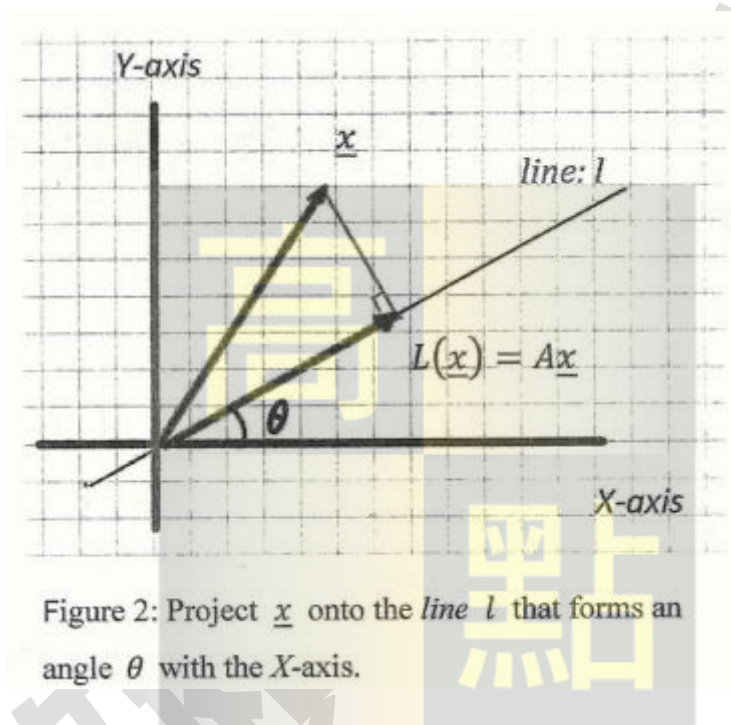
3. Let $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the mapping of a vector $\mathbf{x}$ onto its orthogonal projection on a line ℓ passing through the origin, which forms an angle θ with the x -axis as shown in Figure 2. Let $\mathbf{A}$ be the matrix that corresponds to this mapping, i.e., $L(\mathbf{x}) = \mathbf{Ax}$. Find $\mathbf{A}$ by each method below.

(a) Method 1: According to the formula $\mathbf{A} = [L(\mathbf{e}_1) \ L(\mathbf{e}_2)]$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{A}$ is obtained by evaluating the mappings on the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

(b) Method 2 : Perform a sequential linear transformation

$L(\mathbf{x}) = L_\theta \circ L_x \circ L_{-\theta}(\mathbf{x})$ as described below:

- (i) First, apply $L_{-\theta}$ as clockwise rotation about the origin by θ radians. Determine $\mathbf{A}_1$ such that $L_{-\theta}(\mathbf{x}) = \mathbf{A}_1\mathbf{x}$.
- (ii) Next, carry out L_x as an orthogonal projection of any vector $\mathbf{x}$ onto the x -axis. determine $\mathbf{A}_2$ such that $L_x(\mathbf{x}) = \mathbf{A}_2\mathbf{x}$.
- (iii) Finally, execute a counterclockwise rotation L_θ . Determine $\mathbf{A}_3$ such that $L_\theta(\mathbf{x}) = \mathbf{A}_3\mathbf{x}$. Calculate $\mathbf{A} = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1$. (The result should match the answer given in part(a).)



解答

一、Calculus

$$1. \lim_{x \rightarrow \pi} \left(\csc \frac{x}{2} \right)^{\tan^2 \frac{x}{2}}$$

$$\because \left(\csc \frac{x}{2} \right)^{\tan^2 \frac{x}{2}} = e^{\tan^2 \frac{x}{2} \ln \left(\csc \frac{x}{2} \right)}$$

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi} \tan^2 \frac{x}{2} \ln \left(\csc \frac{x}{2} \right) &= \lim_{x \rightarrow \pi} \frac{\ln \left(\csc \frac{x}{2} \right)}{\cot^2 \frac{x}{2}} = \lim_{x \rightarrow \pi} \frac{\frac{-\csc \frac{x}{2} \cos \frac{x}{2}}{2 \csc \frac{x}{2}}}{-\cot \frac{x}{2} \csc^2 \frac{x}{2}} = \lim_{x \rightarrow \pi} \frac{1}{2 \csc^2 \frac{x}{2}} \\ &= \frac{1}{2} \end{aligned}$$

故原式 = $e^{\frac{1}{2}}$ 。

$$2. \lim_{n \rightarrow \infty} \left(\frac{\pi}{\sqrt{n^2+1}} + \frac{\pi}{\sqrt{n^2+2^2}} + \cdots + \frac{\pi}{\sqrt{n^2+n^2}} + \right)$$

$$\begin{aligned} &= \pi \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{1+\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{1+\left(\frac{2}{n}\right)^2}} + \cdots + \frac{1}{\sqrt{1+\left(\frac{n}{n}\right)^2}} \right) \frac{1}{n} \\ &= \pi \int_0^1 \frac{1}{\sqrt{1+x^2}} dx = \pi \left[\ln \left| x + \sqrt{1+x^2} \right| \right]_0^1 = \pi \ln(1+\sqrt{2})。 \end{aligned}$$

$$3. 2(x^2+y^2)^2 + (x^2-1) = 25(x^2-y^2) + 2^x$$

$$\text{微分得 } 4(x^2+y^2)(2x+2yy') + 2x = 25(2x-2yy') + 2^x \ln 2$$

$$\text{代入 } (3,1) \text{ 得 } 40(6+2y') + 6 = 25(6-2y') + 8 \ln 2$$

$$\rightarrow y' \Big|_{(3,1)} = \frac{-96+8 \ln 2}{130}。$$