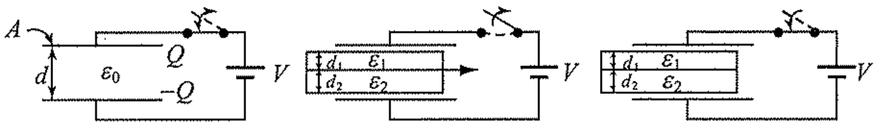
範題 27

A parallel-pate capacitor of area A and separation d is filled with air ε_0, as shown in Fig.(a). Neglect the fringing effects for the following questions.

(1) Using the applied voltage V and accumulated charges Q, derive the capacitance C of such a structure in terms of above parameters.

(2) Repeat (1) and open the switch. Now insert two stacked dielectric slabs of $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 3\varepsilon_0$, $d_1 = \frac{2d}{5}$, $d_2 = \frac{3d}{5}$, as shown in Fig.(b). Find the new capacitance of such a configuration.

(3) Repeat (2) and close the switch, as shown in Fig.(c). What is the change of the voltage across the plates (unchanged, increased or decreased) while turning the switch from 'open' to 'close' state? After closing the switch (steady state), find the capacitance of such a configuration.



(a) (b) (c)

【97台大電信】

【解答】

(1) 設 $\pm Q \to \vec{D} \to \vec{E} \to V$，$C = \dfrac{Q}{V}$

由Gauss Law：$\oint_S \vec{D} \cdot d\vec{S} = Q_{in} \Rightarrow D = \rho_S = \dfrac{Q}{A}$

$$E = \frac{D}{\varepsilon_0} = \frac{\rho_S}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

$$V = -\int_0^d \vec{E} \cdot d\vec{\ell} = \frac{Qd}{A\varepsilon_0}$$

$$\therefore C=\frac{Q}{V}=\varepsilon_0\frac{A}{d}$$

(2) 當switch打開後，此時Q固定，且V將會改變，重覆(1)小題之計算。

$$\oint_S \vec{D}\cdot d\vec{S}=Q_{in}\Rightarrow D=\rho_S=\frac{Q}{A}$$

$$E_1=\frac{D}{\varepsilon_1}=\frac{Q}{A\varepsilon_1}$$

$$E_2=\frac{D}{\varepsilon_2}=\frac{Q}{A\varepsilon_2}$$

$$V'=-\int_0^{\frac{3}{5}d}\vec{E}_2\cdot d\vec{\ell}-\int_{\frac{3}{5}d}^{d}\vec{E}_1\cdot d\vec{\ell}=\frac{Q}{A\varepsilon_1}\cdot\frac{3}{5}d+\frac{Q}{A\varepsilon_2}\cdot\frac{2}{5}d$$

$$V'=Q\left[\frac{d}{A\varepsilon_0}\cdot\frac{3}{10}+\frac{d}{A\varepsilon_0}\cdot\frac{2}{15}\right]=\frac{13}{30}\cdot\frac{dQ}{A\varepsilon_0}$$

$$\therefore C=\frac{Q}{V'}=\frac{30}{13}\frac{A\varepsilon_0}{d}$$

(3) 當switch再度關閉（close）後，由於強加V電壓於電容上，所以，電壓值固定為V，且電容值是由架構所決定，因此平行板兩端之電荷Q須改變。

$$Q'=CV=\frac{30}{13}\frac{A\varepsilon_0}{d}V$$

與(1)小題作比較

$$Q_{(1)}=CV=\varepsilon_0\frac{A}{d}V\Rightarrow Q'-Q_{(1)}=\frac{17}{13}\frac{A\varepsilon_0}{d}V$$（增加之電荷）

範題 28

A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_0. The dielectric constant of the shell is ε_r. Determine $\vec{E}$, V, $\vec{D}$, and $\vec{P}$ as functions of the radial distance $\vec{R}$. 【97台大工科】

【解答】

由Gauss Law：$\oint_S \vec{D}\cdot d\vec{S}=Q_{in}\Rightarrow \vec{D}=\frac{Q}{4\pi R^2}\hat{a}_R$ everywhere.

(1) $R < R_i$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = 0$$

(2) $R_i < R < R_0$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}\hat{a}_R$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = \frac{Q}{4\pi R^2}\left(1 - \frac{1}{\varepsilon_r}\right)$$

(3) $R > R_0$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = 0$$

$$V(R)|_{R>R_0} = -\int_\infty^R \vec{E}\cdot d\vec{R} = \frac{Q}{4\pi\varepsilon_0 R}$$

$$V(R)|_{R_i<R<R_0} = -\int_\infty^{R_0} \vec{E}|_{R>R_0}\cdot d\vec{R} - \int_{R_0}^{R} \vec{E}|_{R_i<R<R_0}\cdot d\vec{R}$$

$$= \frac{Q}{4\pi\varepsilon_0 R_0} + \frac{Q}{4\pi\varepsilon_0\varepsilon_r R} - \frac{Q}{4\pi\varepsilon_0\varepsilon_r R_0} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_0} - \frac{1}{\varepsilon_r R_0} + \frac{1}{\varepsilon_r R}\right]$$

$$V(R)|_{R<R_i} = -\int_\infty^{R_0} \vec{E}|_{R>R_0}\cdot d\vec{R} - \int_{R_0}^{R_i} \vec{E}|_{R_i<R<R_0}\cdot d\vec{R} - \int_{R_i}^{R} \vec{E}|_{R<R_i}\cdot d\vec{R}$$

$$V(R)|_{R<R_i} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_0} - \frac{1}{\varepsilon_r R_0} + \frac{1}{\varepsilon_r R_i} - \frac{1}{R_i} + \frac{1}{R}\right]$$

將上述之各物理量整理如下表：

	$R < R_i$	$R_i < R < R_0$	$R > R_0$
$\vec{D}$	$\frac{Q}{4\pi R^2}\hat{a}_R$	$\frac{Q}{4\pi R^2}\hat{a}_R$	$\frac{Q}{4\pi R^2}\hat{a}_R$
$\vec{E}$	$\frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$	$\frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}\hat{a}_R$	$\frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R$
$\vec{P}$	0	$\frac{Q}{4\pi R^2}\left(1-\frac{1}{\varepsilon_r}\right)$	0

	$R<R_i$	$R_i<R<R_0$	$R>R_0$
V	$\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_0}-\frac{1}{\varepsilon_r R_0}+\frac{1}{\varepsilon_r R_i}-\frac{1}{R_i}+\frac{1}{R}\right]$	$\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_0}-\frac{1}{\varepsilon_r R_0}+\frac{1}{\varepsilon_r R}\right]$	$\frac{Q}{4\pi\varepsilon_0 R}$

範題 29

將自由電荷 q 於一半徑為 a 的導體球上，請問此一電荷分布儲存多少能量？ 【97北科大光電】

【解答】

靜電能 $W=\int_V \frac{1}{2}\varepsilon E^2 dV$

由Gauss Law：$\oint_S \vec{E}\cdot d\vec{S}=\frac{Q_{in}}{\varepsilon}$

$$\begin{cases} R<a\text{區}, & \vec{E}=0 \\ R>a\text{區}, & \vec{E}=\dfrac{q}{4\pi\varepsilon_0 R^2}\hat{a}_R \end{cases}$$

$$\therefore W=\int_V \frac{1}{2}\varepsilon_0\left(\frac{q}{4\pi\varepsilon_0 R^2}\right)^2 dV'=\int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\frac{1}{2}\cdot\left(\frac{q}{4\pi R^2}\right)^2 R^2\sin\theta dRd\theta d\phi$$

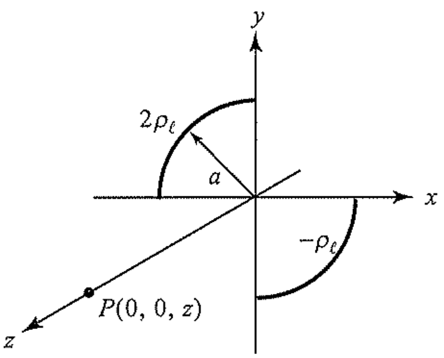
$$=\int_a^{\infty}\frac{1}{2}\cdot\left(\frac{q}{4\pi R^2}\right)^2\cdot 4\pi R^2 dR=\frac{q^2}{8\pi a}$$

範題 30

For two quarter circular line charges of density $2\rho_\ell$ and $-\rho_\ell$, respectively, located on the x-y plane, as shown in Fig. below, determine the following quantities at any point $(0,0,z)$ on the z-axis,

(1) the electric potential V.

(2) the electric field intensity $\vec{E}$.



【97北科大電通】

此為非對稱架構，須由庫侖定律解 $\vec{E}$。

【解答】

(1) $V=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_C\frac{\rho_\ell d\ell}{R}=\frac{1}{4\pi\varepsilon_0}\left[\int_{-\frac{\pi}{2}}^{0}\frac{-\rho_\ell}{(z^2+a^2)^{\frac{1}{2}}}ad\phi+\int_{\frac{\pi}{2}}^{\pi}\frac{2\rho_\ell}{(z^2+a^2)^{\frac{1}{2}}}ad\phi\right]$

$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{-a\rho_\ell}{(z^2+a^2)^{\frac{1}{2}}}\cdot\frac{\pi}{2}+\frac{2a\rho_\ell}{(z^2+a^2)^{\frac{1}{2}}}\cdot\frac{\pi}{2}\right]=\frac{a\rho_\ell}{8\varepsilon_0(z^2+a^2)^{\frac{1}{2}}}$$

(2)庫侖定律：

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{\ell'}\frac{\rho_\ell d\ell'(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

其中 $\vec{r}=z\hat{a}_z$

$\vec{r}'=a\hat{a}_\phi$

$|\vec{r}-\vec{r}'|^3=(z^2-a^2)^{\frac{3}{2}}$

$d\ell'=ad\phi$

代回庫侖定律

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\left[\int_{-\frac{\pi}{2}}^{0}\frac{-\rho_\ell(z\hat{a}_z-a\hat{a}_\phi)}{(z^2-a^2)^{\frac{3}{2}}}ad\phi+\int_{\frac{\pi}{2}}^{\pi}\frac{2\rho_\ell(z\hat{a}_z-a\hat{a}_\phi)}{(z^2-a^2)^{\frac{3}{2}}}ad\phi\right]$$

由於 $\hat{a}_\phi$ 爲積分路徑 $d\phi$ 之函數，因此無法直接積分，須分爲 $\hat{a}_x$，$\hat{a}_y$，$\hat{a}_z$ 三分量。

$$\vec{E}=E_x\hat{a}_x+E_y\hat{a}_y+E_z\hat{a}_z$$

其中 $\begin{cases}E_x=\vec{E}\cdot\hat{a}_x\\E_y=\vec{E}\cdot\hat{a}_y\\E_z=\vec{E}\cdot\hat{a}_z\end{cases}$

$$E_z=\vec{E}\cdot\hat{a}_z=\frac{1}{4\pi\varepsilon_0}\left[\frac{-\rho_\ell za}{(z^2-a^2)^{\frac{3}{2}}}\cdot\frac{\pi}{2}+\frac{2\rho_\ell za}{(z^2-a^2)^{\frac{3}{2}}}\cdot\frac{\pi}{2}\right]=\frac{\rho_\ell za}{8\varepsilon_0(z^2-a^2)^{\frac{3}{2}}}$$

$$E_x=\vec{E}\cdot\hat{a}_x=\frac{1}{4\pi\varepsilon_0}\left[\int_{-\frac{\pi}{2}}^{0}\frac{\rho_\ell a^2}{(z^2-a^2)^{\frac{3}{2}}}\hat{a}_x\cdot\hat{a}_\phi d\phi+\int_{\frac{\pi}{2}}^{\pi}\frac{-2\rho_\ell a^2}{(z^2-a^2)^{\frac{3}{2}}}\hat{a}_x\cdot\hat{a}_\phi d\phi\right]$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\int_{-\frac{\pi}{2}}^{0}\frac{\rho_\ell a^2}{(z^2-a^2)^{\frac{3}{2}}}(-\sin\phi)d\phi+\int_{\frac{\pi}{2}}^{\pi}\frac{-2\rho_\ell a^2}{(z^2-a^2)^{\frac{3}{2}}}(-\sin\phi)d\phi\right]$$

$$=\frac{3\rho_\ell a^2}{4\pi\varepsilon_0(z^2-a^2)^{\frac{3}{2}}}$$

$$E_y=\vec{E}\cdot\hat{a}_y=\frac{1}{4\pi\varepsilon_0}\left[\int_{-\frac{\pi}{2}}^{0}\frac{\rho_\ell a^2}{(z^2-a^2)^{\frac{3}{2}}}\cos\phi d\phi+\int_{\frac{\pi}{2}}^{\pi}\frac{-2\rho_\ell a^2}{(z^2-a^2)^{\frac{3}{2}}}\cos\phi d\phi\right]$$

$$=\frac{-3\rho_\ell a^2}{4\pi\varepsilon_0(z^2-a^2)^{\frac{3}{2}}}$$

範題 31

A spherical dielectric shell of an inner radius r_1 and an outer radius r_0 is centered at the origin and has a dielectric constant of ε_r. Given a charge distribution

$$\rho_v\ (C/m^3)=\begin{cases}\rho_0\left(1-\dfrac{r^2}{r_i^2}\right) & r<r_i\\ 0 & \text{else}\end{cases}$$, where $r=\sqrt{x^2+y^2+z^2}$, determine

(1) $\vec{E}$ in $0\le r\le r_0$.

(2) V and $\vec{P}$ inside the dielectric shell.　【97中山電機、通訊】

【解答】

(1)由Gauss Law：$\oint_S\vec{E}\cdot d\vec{S}=\dfrac{Q_{in}}{\varepsilon}$

① $0\le r\le r_i$ 區：

$$E\cdot 4\pi r^2=\frac{1}{\varepsilon_0}\int_V\rho_V dV=\frac{1}{\varepsilon_0}\int_0^r\rho_0\left(1-\frac{r^2}{r_i^2}\right)4\pi r^2dr=\frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{r^3}{3}-\frac{r^5}{5r_i^2}\right]$$

$$\therefore\vec{E}=\frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3}-\frac{r^3}{5r_i^2}\right]\hat{a}_R$$

② $r_i<r<r_0$：

$$E\cdot 4\pi r^2=\frac{1}{\varepsilon_0\varepsilon_r}\int_V\rho_V dV=\frac{4\pi\rho_0}{\varepsilon_0\varepsilon_r}\left[\frac{r_i^3}{3}-\frac{r_i^3}{5}\right]$$

$$\therefore \vec{E} = \frac{8\rho_0 r_i^3}{15\varepsilon_0\varepsilon_r r^2}\hat{a}_R$$

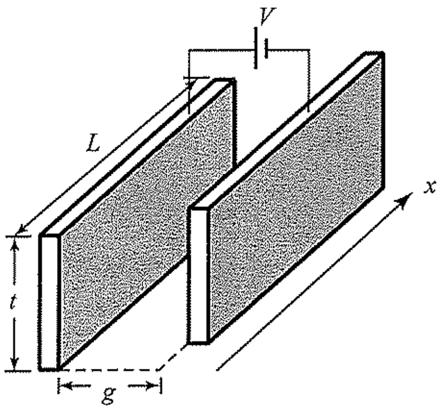
(2) $\vec{D} = \varepsilon\vec{E} = \dfrac{8\rho_0 r_i^3}{15r^2}\hat{a}_R$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = \frac{8\rho_0 r_i^3}{15r^2}\left(1 - \frac{1}{\varepsilon_r}\right)\hat{a}_R$$

$$V(r)\big|_{r_i<r<r_0} = -\int_{\infty}^{r_0}\frac{8\rho_0 r_i^3}{15\varepsilon_0 r^2}dr - \int_{r_0}^{r}\frac{8\rho_0 r_i^3}{15\varepsilon_0\varepsilon_r r^2}dr = \frac{8\rho_0 r_i^3}{15\varepsilon_0 r_0} - \frac{8\rho_0 r_i^3}{15\varepsilon_0\varepsilon_r r_0} + \frac{8\rho_0 r_i^3}{15\varepsilon_0\varepsilon_r r}$$

範題 32

Figure shows two conductive plates which are parallel with each other. Each plate has a height t and a length L. The gap between them is g. The right plate is mechanically fixed, and the left plate is free to slide only in the $\pm x$ directions. With a constant voltage V applied across the two plates, the movable left plate experiences a mechanical force of electric origin. Consider the left plate is located closer to you as depicted by the figure. Derive $F_{e,x}$ the x component of that force. Be careful with the "+" and/or "−" sign(s) in your answer.



[Note: Ignore the fringing of fields at the edges of the plates.]

【96台大光電】

【解答】

利用虛位移法解靜電力，此為電位固定系統 $\vec{F}_V$。

$$\vec{F}_V = \frac{1}{2}V^2\nabla C$$