

$$G.M. = 20 \log \frac{1}{2} = -6\text{dB}$$

若  $C(s)G(s)$  為極小相位，則閉迴路系統為穩定；但若  $C(s)G(s)$  為非極小相位，則閉迴路系統之穩定性將不能只以增益邊際  $G.M.$  判定之。

### 題型9-23 → 增益邊際與相位邊際之計算

#### •範題 30 •



Given a feedback system as shown in Fig. 2, where  $r$  is the reference input,  $y$  is the output,  $e$  is the error.  $K$  is a constant gain.

The plant  $G(s)$  is given as:

$$G(s) = \frac{20}{s(s+2)(s+10)}$$

The frequency response of  $G(s)$  is given as Fig. 3.

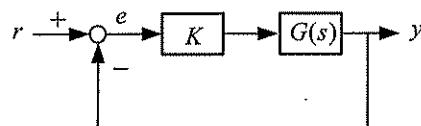


Fig. 2

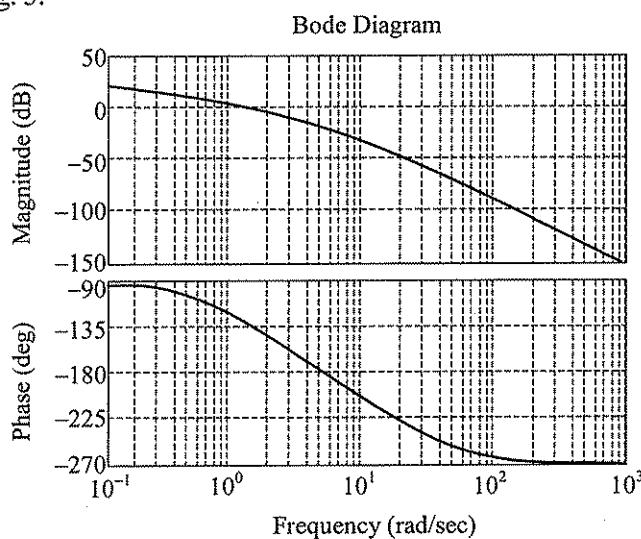


Fig. 3

9-54 自動控制系統經典題型

- (1) Please determine the frequency  $\omega_{GM}$  analytically where the phase of  $G(j\omega_{GM})$  is equal to  $-180^\circ$ .
- (2) Plot the Nyquist Diagram of  $G(s)$ .
- (3) Find the stable range of  $K$  for the closed loop system between  $r$  and  $y$  using the information obtained from (1) and (2). Find the gain margin.
- (4) Explain why any  $K$  smaller than 0 will be unstable using Nyquist criterion.
- (5) Find the approximate value of the crossover frequency  $\omega_c$  from Fig. 3. Use the  $\omega_c$  to calculate the phase margin of the system.
- (6) Plot the approximate step response of  $y(t)$ . You should identify the approximate rise time, 2% settling time, overshoot, and steady state error to get the full credits.

(中正機械)

**【解析】**

- (1) 頻率響應函數  $G(j\omega)$  為

$$G(j\omega) = \frac{20}{(j\omega)(j\omega+2)(j\omega+10)} \\ = \frac{20}{\omega\sqrt{\omega^2 + 4\sqrt{\omega^2 + 100}}} \quad \left| -90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10} \right.$$

依定義知

$$\angle G(j\omega_{GM}) = -180^\circ = -90^\circ - \tan^{-1}\frac{\omega_{GM}}{2} - \tan^{-1}\frac{\omega_{GM}}{10} \\ \Rightarrow \tan^{-1}\frac{\omega_{GM}}{2} + \tan^{-1}\frac{\omega_{GM}}{10} = 90^\circ \\ \Rightarrow \frac{\frac{\omega_{GM}}{2} + \frac{\omega_{GM}}{10}}{1 - \frac{\omega_{GM}}{2} \times \frac{\omega_{GM}}{10}} = \tan 90^\circ = \infty \\ \Rightarrow 1 - \frac{\omega_{GM}^2}{20} = 0$$

可解得  $\omega_{GM} = 4.47\text{rad/s}$

- (2) ① 選擇奈氏路徑  $\Gamma_s$  如圖(a)所示，因奈氏路徑  $\Gamma_s$  並未包圍  $G(s)$  之極點，

所以  $P = 0$ 。

② 將奈氏路徑  $\Gamma_s$  對  $G(s)$  映射，分段處理如下

$\overline{ab}$  段： $s = j\omega, \omega = 0^+ \rightarrow +\infty$

$$G(j\omega) = \frac{20[-12\omega + j(\omega^2 - 20)]}{\omega(4 + \omega^2)(100 + \omega^2)}$$

$$G(j0^+) = \infty \angle -90^\circ$$

$$G(j\infty) = 0 \angle -270^\circ$$

令  $\text{Im}[G(j\omega)] = 0 \Rightarrow \omega = \sqrt{20} \text{ rad/s}$ ，所以奈氏圖  $\Gamma_G$  與實軸交

$$\text{於 } \text{Re}[G(j\omega)] = \frac{-20 \times 12 \times \sqrt{20}}{\sqrt{20}(4 + 20)(100 + 20)} = -0.083$$

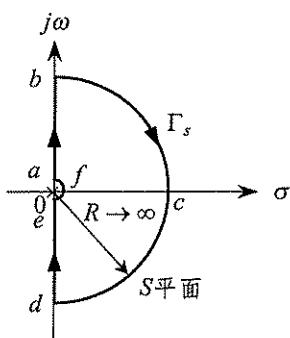
$\widehat{bcd}$  段映射於  $G(s)$  平面上之原點，而  $\overline{de}$  段之映射與  $\overline{ab}$  段之映射對稱於  $G(s)$  平面之實軸。

$\widehat{efa}$  段： $s = \lim_{\rho \rightarrow 0} \rho e^{j\theta}, \theta = -90^\circ \rightarrow 0^\circ \rightarrow +90^\circ$

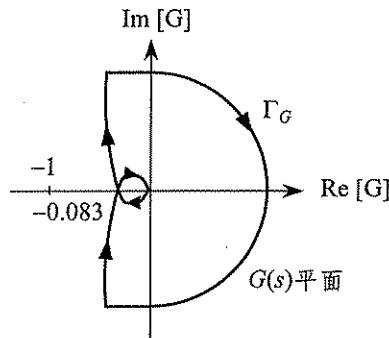
$$G(s) = \lim_{\rho \rightarrow 0} \frac{20}{\rho e^{j\theta} (\rho e^{j\theta} + 2)(\rho e^{j\theta} + 10)} = \lim_{\rho \rightarrow 0} \frac{1}{2\rho e^{j\theta}}$$

$$= \infty \angle -\theta = \begin{cases} \infty \angle +90^\circ & (\theta = -90^\circ) \\ \infty \angle 0^\circ & (\theta = 0^\circ) \\ \infty \angle -90^\circ & (\theta = +90^\circ) \end{cases}$$

綜合以上結果，可繪出奈氏圖  $\Gamma_G$  如圖(b)所示。



圖(a)



圖(b)

(3) 因為  $P = 0$ ，由奈氏穩定性定理知系統為穩定之條件為  $N = -P = 0$ ，亦

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即奈氏圖  $\Gamma_G$  不能包圍點  $(-1, 0)$ ，因此

$$-1 < -0.083 \times K$$

可解得  $0 < K < 12$ 。

(4)若  $K < 0$ ，則奈氏圖將與  $K > 0$  之奈氏圖對稱於原點，因此  $K < 0$  之奈氏圖必會包圍點  $(-1, 0)$ ，所以  $N = -P = 0$  必不會成立，故系統必為不穩定。

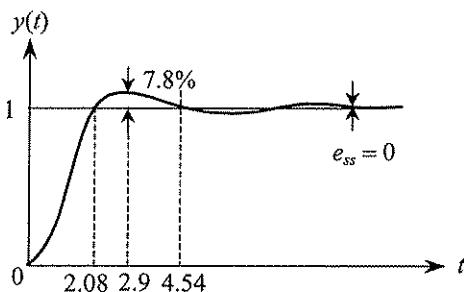
(5)由圖中可看出  $\omega_c = 0.9\text{rad/s}$ ，所以相位邊際  $P.M.$  為

$$P.M. = 180^\circ + \left( -90^\circ - \tan^{-1} \frac{0.9}{2} - \tan^{-1} \frac{0.9}{10} \right) = 60.64^\circ$$

(6)考慮  $K = 1$ ，則閉迴路轉移函數  $G_c(s)$  為

$$\begin{aligned} G_c(s) &= \frac{20}{s(s+2)(s+10)} \\ &= \frac{20}{1 + \frac{s(s+2)(s+10)}{20}} \\ &= \frac{20}{s^3 + 12s^2 + 20s + 20} \\ &= \frac{20}{(s+10.24)(s^2 + 1.76s + 1.95)} \\ &\approx \frac{1.95}{s^2 + 1.76s + 1.95} \end{aligned}$$

所以  $2\zeta\omega_n = 1.76$ ,  $\omega_n^2 = 1.95$ ，可解得  $\zeta = 0.63$ ,  $\omega_n = 1.4\text{rad/s}$ ，因此可求得上升時間  $t_r = 2.08\text{sec}$ ，2% 安定時間  $t_s = 4.54\text{sec}$ ，最大超越量  $M_p = 0.078 = 7.8\%$ ，而因為系統為一型（Type 1），所以穩態誤差  $e_{ss} = 0$ 。由以上結果可繪出步階響應如圖(c)所示。



圖(c)