113 成大土木所

科目：工數

🕮題目

1. Consider the following second-order linear nonhomogeneous differential equation with constant coefficients: $y''+4y'-5y=10te^{-t}$

   (a)Find the solution to the corresponding homogeneous equation.

   (b)Find a particular solution to the nonhomogeneous equation.

   (c)State the general solution to the differential equation.

2. Given the matrix: $\mathbf{A}=\begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$, $n=9$. Perform the following tasks:

   (a)Determine whether the matrix $\mathbf{A}$ has an eigenvalue λ_1 of multiplicity two.

   (b)Assuming λ_1 is an eigenvalue of multiplicity two, explain why the equation $\lambda^n=c_0+c_1\lambda$ does not yield enough independent equations to solve for the coefficients c_i.

   (c)Show how to use the derivative to of the eigenvector equation evaluated at λ_1 to obtain an extra equation needed to form a complete system.

   (d)Compute $\mathbf{A}^n$ and use this result to compute the indicated power of the matrix $\mathbf{A}$.

3. Given the scalar field $g=e^{x^2+y^2+z^2}$ and the vector field $\vec{v}=yz\vec{i}+xz\vec{j}+xy\vec{k}$, find the following:

   (1) $\operatorname{div}(g\vec{v})$

   (2) $\nabla^2 g$

   (3) $\operatorname{curl}(\operatorname{grad} g)$

   (4) $\operatorname{div}(\operatorname{curl} \vec{v})$

   (5) $\operatorname{grad}(\vec{v}\cdot\vec{v})$

4. Find the Fourier integral representation of the piecewise-continuous function:

$$f(x)=\begin{cases}0\,, & x<0\\ 1\,, & 0<x<3\\ 0\,, & x>3\end{cases}$$

5. Find an LU-factorization of $\begin{bmatrix}3 & 3 & -6\\ 2 & -2 & 4\\ 1 & 1 & -2\end{bmatrix}$.

☞解答

1. ODE： $y''+4y'-5y=10te^{-t}$

(a)令 $y_h=e^{\lambda t}$ 代入得 $\lambda^2+4\lambda-5=0 \Rightarrow \lambda=1,-5$

$\therefore y_h(t)=c_1e^{t}+c_2e^{-5t}$ 。

(b)令 $y_p=(At+B)e^{-t}$ 代入比較係數得 $A=-\dfrac{5}{4},\ B=-\dfrac{5}{16}$

故 $y_p(t)=\left(-\dfrac{5}{4}t-\dfrac{5}{16}\right)e^{-t}$ 。

(c) $y(t)=y_h+y_p=c_1e^{t}+c_2e^{-5t}+\left(-\dfrac{5}{4}t-\dfrac{5}{16}\right)e^{-t}$ 。

2. $\mathbf{A}=\begin{bmatrix}8 & 3\\ -3 & 2\end{bmatrix},\ n=9$

(a) $|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}8-\lambda & 3\\ -3 & 2-\lambda\end{vmatrix}=(\lambda-5)^2=0$

$\therefore \lambda_1=\lambda_2=5$～二重根。

(b)因此 $\lambda^9=c_0+c_1\lambda$

代入 $\lambda_1=5$ 僅能得到 $5^9=c_0+5c_1$，無法解出 c_i。

(c)對 $\lambda^9=c_0+c_1\lambda$ 微分得 $9\lambda^8=c_1$。

(d)由 $\begin{cases}\lambda^9=c_0+c_1\lambda\\ 9\lambda^8=c_1\end{cases}$，代入 $\lambda=5$ 得 $\begin{cases}5^9=c_0+5c_1\\ 9\cdot5^8=c_1\end{cases}$

解得 $c_0=-8\cdot5^9,\ c_1=9\cdot5^8$

$\therefore \lambda^9=-8\cdot5^9+9\cdot5^8\lambda$

因此 $\mathbf{A}^9=-8\cdot5^9\mathbf{I}+9\cdot5^8\mathbf{A}=-8\cdot5^9\begin{bmatrix}1&0\\0&1\end{bmatrix}+9\cdot5^8\begin{bmatrix}8&3\\-3&2\end{bmatrix}$

$=\begin{bmatrix}32\cdot5^8&27\cdot5^8\\-27\cdot5^8&-22\cdot5^8\end{bmatrix}$。

3. $g=e^{x^2+y^2+z^2}$，$\vec{v}=yz\vec{i}+xz\vec{j}+xy\vec{k}$

(1) $\operatorname{div}(g\vec{v})=g\nabla\cdot\vec{v}+\vec{v}\cdot\nabla g$

$=e^{x^2+y^2+z^2}\cdot0+(yz\vec{i}+xz\vec{j}+xy\vec{k})\cdot e^{x^2+y^2+z^2}(2x\vec{i}+2y\vec{j}+2z\vec{k})$

$=6xyze^{x^2+y^2+z^2}$。

(2) $\nabla^2 g=\left[6+4\left(x^2+y^2+z^2\right)\right]e^{x^2+y^2+z^2}$。

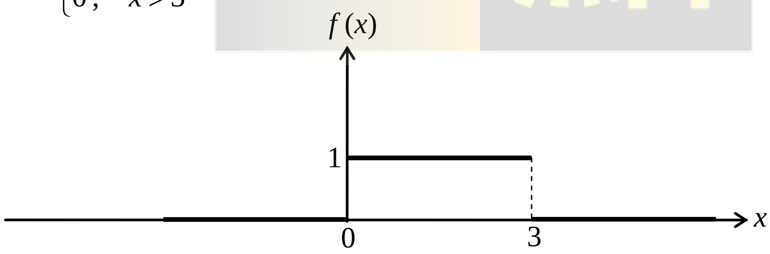
(3) $\operatorname{curl}(\operatorname{grad} g)=0$。(恆等式)

(4) $\operatorname{div}(\operatorname{curl}\vec{v})=0$。(恆等式)

(5) $\operatorname{grad}(\vec{v}\cdot\vec{v})=\nabla(y^2z^2+x^2z^2+x^2y^2)$

$=2x(y^2+z^2)\vec{i}+2y(x^2+z^2)\vec{j}+2z(x^2+y^2)\vec{k}$。

4. $f(x)=\begin{cases}0, & x<0\\ 1, & 0<x<3\\ 0, & x>3\end{cases}$



$$f(x)=\int_0^{\infty}\left[A(\omega)\cos\omega x+B(\omega)\sin\omega x\right]d\omega$$

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\cos\omega x dx=\frac{1}{\pi}\int_0^3\cos\omega x dx=\frac{\sin(3\omega)}{\pi\omega}$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\sin\omega x dx=\frac{1}{\pi}\int_0^3\sin\omega x dx=\frac{1-\cos(3\omega)}{\pi\omega}$$

故 $f(x)=\int_0^{\infty}\left[\frac{\sin(3\omega)}{\pi\omega}\cos\omega x+\frac{1-\cos(3\omega)}{\pi\omega}\sin\omega x\right]d\omega$ 。

5. $\mathbf{A}=\begin{bmatrix}3 & 3 & -6\\ 2 & -2 & 4\\ 1 & 1 & -2\end{bmatrix}$

$$\begin{bmatrix}3 & 3 & -6\\ 2 & -2 & 4\\ 1 & 1 & -2\end{bmatrix}\xrightarrow{Gauss}\begin{bmatrix}3 & 3 & -6\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}\equiv\mathbf{U}$$

即 $\underbrace{\begin{bmatrix}1 & 0 & 0\\ -\frac{2}{3} & 1 & 0\\ -\frac{1}{3} & 0 & 1\end{bmatrix}}_{\mathbf{E}_1}\underbrace{\begin{bmatrix}3 & 3 & -6\\ 2 & -2 & 4\\ 1 & 1 & -2\end{bmatrix}}_{\mathbf{A}}=\underbrace{\begin{bmatrix}3 & 3 & -6\\ 0 & -4 & 8\\ 0 & 0 & 0\end{bmatrix}}_{\mathbf{U}}$

所以 $\mathbf{E}_1=\begin{bmatrix}1 & 0 & 0\\ -\frac{2}{3} & 1 & 0\\ -\frac{1}{3} & 0 & 1\end{bmatrix}\rightarrow\mathbf{E}_1^{-1}=\begin{bmatrix}1 & 0 & 0\\ \frac{2}{3} & 1 & 0\\ \frac{1}{3} & 0 & 1\end{bmatrix}$ （看清楚了嗎?）

$\therefore\ \mathbf{A}=(\mathbf{E}_1)^{-1}\mathbf{U}\equiv\mathbf{LU}$

$\therefore\ \mathbf{L}=(\mathbf{E}_1)^{-1}=\mathbf{E}_1^{-1}=\begin{bmatrix}1 & 0 & 0\\ \frac{2}{3} & 1 & 0\\ \frac{1}{3} & 0 & 1\end{bmatrix}$ 。

本份解完！